A (very, very) deep dive into the computation of Elevation Scale Factors / Ellipsoidal Reduction for GRS80

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In my previous blog/FAQ entry I noted that there will be a variety of ESF (Elevation Scale Factors) to be encountered for a given latitude and ellipsoid height.

The previous FAQ's example for this ground system base position:

Latitude	37 07 48.88043 (DMS.s)	37.1302445639 (D.d)
Ellipsoid Height	2706.894 (usfeet)	825.063 (m)

presented three values from NGS NCAT, Carlson and the simplified method that I have used in the past.

In addition, if you poke around the web looking for 'calculators of earth radius at a latitude', or investigate ground systems computed by survey packages, you can find several online calculators, values and methods. All give significantly different values. For example:

NGS:	6,372,281.171 meters	
Retchner:	6,370,387	(https://rechneronline.de/earth-radius/)
PlanetCalc:	6,378,137	(https://planetcalc.com/7721/)

Some of the values assume the WGS84 ellipsoid, others have completely different computations in play.

Additionally the NGS has traditionally suggested a 'good-enough' approximation, in feet:

$$ESF = \frac{20906000}{2090600+H} = \frac{20906000}{2090600+2706.894} = 0.999,870,537,474,473$$

Since the GRS80 ellipsoid is used by NAD83 and the future (2024?) SPC zones, it may be worth a few minutes to compare these differences and see if they have any affect on scaled distances.

For the purposes of survey work in the USA, let's assume that the NGS NCAT result is correct.

Comparing Some Resources

1. From NCAT:

Reference Frame:NAD83(2011)

Lat-Lon-Height		SPC		UTM/USNG		XYZ (m)	
Latitude	N37° 07' 48.88043" N370748.88043 37.1302445639	Zone	UT S-4303	Zone	12	X	-2,031,208.418
Longitude	E246° 29' 24.55035" W1133035.44965 -113.5098471250	Northing	3,053,368.770 (m) 10,017,594.041 (usft) 10,017,614.076 (ift)	Northing (m)	4,112,269.745	Y	-4,669,264.867
Ellipsoid Height (m)	825.063	Easting	321,416.652 (m) 1,054,514.467 (usft) 1,054,516.576 (ift)	Easting (m)	277,045.630	Z	3,829,425.087
		Convergence (dms)	-01 13 53.07	Convergence (dms)	-01 30 56.31		
		Scale factor	1.00001594	Scale factor	1.00021239		
		Combined factor	0.99988648	Combined factor	1.00008290		
				USNG	12STG7704612270		


```
Grid SF:                 1.000015941378
Combined SF:             0.99988648
```

thus

```
Elevation SF = CSF/GSF:
                         0.999,870,54          (computed)
```

Later in this document we will implement the NCAT method to find the value to higher precision:

0.999,870,539,895,356

2. From Carlson SurvCE (Version 6.08):

```
Elevation SF =           0.999,870,514,005
```

3. My traditional 'simplified' method:

$$r_\theta = \frac{a\sqrt{1-e^2}}{1-e^2\sin^2\theta}$$

$$ESF = \frac{r_\theta}{r_\theta + H}$$

$H = Ellipsoid\ Height\ (meters)$
$a = 6378137.0$
$\theta = Latitude\ (radians)$
$e^2 = 0.0066943800229034$

```
a   := 6378137.0; // earth radius at equator
e2  := 0.0066943800229034;  // From Snyder
// Sin of Theta, convert Lat from degrees to radians
st  := sin( DegToRad(Lat) );
// Geometric Mean Radius of Curvature
Rg  := a * sqrt( 1.0 - e2 ) / (1.0 - e2 * st * st );
// Ellipsoidial Reduction
esf := Rg / (Rg + ElevM );
```

which results in:

```
ESF:                     0.999,870,540,191,020
```

4. The NGS Approximation:

$$ESF = \frac{20906000}{2090600+H} = \frac{20906000}{2090600+2706.894} =$$

0.999,870,537,474,473

Summarizing the elevation scale factors and the distance for a 10-mile vector resulting from the Combined Scale Factor using these ESF:

Distance	52800.000	10	miles	
Method	**Elvation SF**	**Grid SF**	**Combined SD**	**Scaled distance**
1: NCAT	0.999870539895356	1.000015941378	0.999886479209584	52805.994578
2: Carlson	0.999870514005000	1.000015941378	0.999886453318815	52805.995946
3: Mark#1	0.999870540191020	1.000015941378	0.999886479505252	52805.994563
4. NGS Approx	0.999870537474473	1.000015941378	0.999886476788662	52805.994706
			Range	0.001383

So, let us be clear here: for most applications the obsessions of this FAQ are pedantry. Applied over a 10-mile distance, the computed ground distances make less a 0.002-foot difference.

However, these computations can also be applied at the origin of a State Plane Coordinate system to a project raised to ground. In the previous FAQ, the distance from the origin to the ground system base is about 1,202,538 feet and the difference between coordinates derived using disparate scale factors for this example is nearly 0.04':

Distance	1202538.000	10	miles	
Method	**Elvation SF**	**Grid SF**	**Combined SD**	**Scaled distance**
1: NCAT	0.999870539895356	1.000015941378	0.999886479209584	1202674.528563
2: Carlson	0.999870514005000	1.000015941378	0.999886453318815	1202674.559705
3: Mark#1	0.999870540191020	1.000015941378	0.999886479505252	1202674.528207
4. NGS Approx	0.999870537474473	1.000015941378	0.999886476788662	1202674.531475
			Range	0.031497

Furthermore, this difference increases with the project elevation, so for high elevations it can exceed a few tenths of a foot.

This can be annoying when comparing ground coordinates computed from grid coordinates between software products like desktop and field tools.

What about a, e^2 and 1/*f*?

The earth equatorial radius for GRS80 is easy as it is defined as:

6,378,137.0 meters

exactly.

A computed value for e^2 to 16-decimal places is presented by Stern in NOAA Manual NOS NGS5, State Plane Coordinate System of 1983 by James E Stern, 1989:

a = 6,378,137. m (exact by definition)

1/f = 298.25722210088 (to 14 significant digits by computation)

From these two numbers, any other desired constants of geometric geodesy may be derived. For example, to 14 significant digits:

b = 6,356,752.3141403

e^2 = 0.0066943800229034.

iGage

```
e^2    =    0.006,694,380,022,903,4
```

However, 1/*f* is the defined value and e^2 is a derived constant. Stern cites a *computed* 14-digit flattening ratio of:

```
1/f    =    298.257,222,100,88
```

The complete definition of GRS80 can be found in the original definition document '*Geodetic Reference System 1980*', H. Moritz (available here), however it does not list a higher resolution value.

A higher precision value for the flattening ratio is listed at Wikipedia:

```
 1/f  =    298.257,222,100,882,711,243
```


Which includes this iterative, very high resolution, solution for *f*:

Derived physical constants (rounded)

Period of rotation (sidereal day) = $2\pi/\omega$ = 86 164.100 637 s

The formula giving the eccentricity of the GRS80 spheroid is:[1]

$$e^2 = \frac{a^2 - b^2}{a^2} = 3J_2 + \frac{4}{15}\frac{\omega^2 a^3}{GM}\frac{e^3}{2q_0},$$

where

$$2q_0 = \left(1 + \frac{3}{e'^2}\right)\arctan e' - \frac{3}{e'}$$

and $e' = \frac{e}{\sqrt{1-e^2}}$ (so $\arctan e' = \arcsin e$). The equation is solved iteratively to give

$$e^2 = 0.00669\,43800\,22903\,41574\,95749\,48586\,28930\,62124\,43890\ \ldots$$

which gives

$$f = 1/298.25722\,21008\,82711\,24316\,28366\ \ldots.$$

There is a great, heated, discussion at SurveyorConnect:

https://surveyorconnect.com/community/gnss-geodesy/grs-80-not-merely-geometric/

that is definitely worth a read. I personally always enjoy a vigorous debate over mathematics

Sensitivity to 1/*f*

When computing the ESF for a latitude and ellipsoid height, e^2 is the numerically significant factor in the computation.

It is interesting to note that any value of 1/*f* between:

```
                123 456 789 012 345
 From 1/f  = 298.257,222,100,881,183
 To   1/f  = 298.257,222,100,885,646
```

Will result in the same e^2 value as Stern's to 16 places:

```
              123 456 789 012 345 6
 E^2  =    0.006,694,380,022,903,4
```

Other values for 1/*f* in common use

It does not take much web searching to find several additional common values for the GRS80 flattening radio.

Perhaps the most common 1/f value is:

```
1/f    =    298.257,222,101
```

Which is the Stern value rounded to 12-decimal places. This 12-digit value is used by the NGS NCAT program and some GIS platforms.

I have also found that this GRS80 flattening radio:

```
1/f =     298.257,222,101,004
```

Which is in VERY common use by various GIS and Survey applications. Do a Google search on '298.257222101004' for use examples. I suspect that this value is a loose computation from e^2.

Do these various 1/*f* values make any difference?

As you may suspect: **No, to a reasonable level they are all the same.**

Here is a comparison of the e^2 values and the resulting Elevation Scale Factors:

	1/*f*	e^2	ESF
a. NCAT	1.0/298.257222101	0.006694380022900780	0.999870539895356
b. Stern NGS NOS5	1.0/298.25722210088	0.006694380022903470	0.999870539895356
c. Stern e^2		0.006694380022903400	0.999870539895356
d. LDO's	1.0/298.257222101004	0.006694380022900690	0.999870539895356
e. NGS Approx			0.999870537474473

The resulting Elevation Scale Factors, to 15-digits are exactly the same.

NCAT / OPUS computation of ESF

The NGS has placed the source code for NCAT online at GitHub. So, we can see what constants and equations they are using in their tool sets.

In the Util.java (https://github.com/noaa-ngs/ncat-lib/blob/main/src/gov/noaa/ngs/transform/Util.java) starting on line 19:

```
public static final double NAD83_RF = 298.257222101; //reciprocal of earth flattening
public static final double NAD83_RADIUS = 6378137.000000; //earth eq. radius
```

Then on line 309 of CoordinateTransformations.java (https://github.com/noaa-ngs/ncat-lib/blob/main/src/gov/noaa/ngs/transform/CoordinateTransformation.java):

```
  * computes elevation factor as a function of lat and ellipsoid height
  *
  * @param lat latitude
  * @param ellipHeight ellipsoid height
  * @return elevation factor
  */
public double getElevationFactor(double lat, double ellipHeight) {
    lat *= degreesToRadians;
    double tmp = Math.sqrt(1.0 - esq * Math.sin(lat) * Math.sin(lat));
    double tmp3 = Math.pow(tmp, 3);
    double n = radius / tmp;
    double m = radius * (1.0 - esq) / tmp3;
    double ra = 2 * m * n / (m + n);
    return ra / (ra + ellipHeight);
}
```

$$r_\theta = \frac{2\dfrac{a}{\sqrt{1-e^2\sin^2\theta}}\,\dfrac{a(1-e^2)}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}}{\dfrac{a}{\sqrt{1-e^2\sin^2\theta}}+\dfrac{a(1-e^2)}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}}$$

'Simplifying':

$$r_\theta = \frac{2a^2(1-e^2)}{(1-e^2\sin^2\theta)^2\left(\frac{a}{\sqrt{1-e^2\sin^2\theta}} + \frac{a(1-e^2)}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}\right)}$$

$$ESF = \frac{r_\theta}{r_\theta + H}$$

$$H = Ellipsoid\ Height\ (meters)$$
$$a = 6378137.0$$
$$\theta = Latitude\ (radians)$$
$$e^2 = 0.0066943800229034$$

I translated this java method to Delphi as:

```
function getNGSElevationFactor( Lat: extended; ElevM: extended;
                              var Rg: extended
                                ): extended;
 var
   sl: extended; // sin(degtorad(lat))
   tmp, tmp3, n, m: extended; // temporary values
 const
   // GRS80
   // define as constants, precompile values and insure stored as 80-bit extended
   a : extended = 6378137.0;             // GRS80 Earth Equatorial Radius
   f = 1.0/298.25722210088;              // inverse flattening radio Stern NOS5
   e2: extended = (f * (2.0 - f));       // Compute e^2, eccentricity squared
 begin
   // cache sin(...
   sl     := sin( DegToRad(Lat) );  // sin(lat) where lat is degrees
   tmp    := sqrt(1.0 - e2 * sl * sl);
   tmp3   := power( tmp, 3 );
   n      := a / tmp;
   m      := a * (1.0 - e2) / tmp3;
   rg     := 2.0 * m * n / (m + n); // earth radius at Lat
   result := rg / (rg + ElevM );    // Elevation scale factor at ElevM
 end;
```

For the example at hand, this method returns:

```
      ESF   =     0.999,870,539,895,356
```

Essentially, an exact match for NCAT.

Differences: WGS84 vs. GRS80

I suspect that it is fairly common for field software and users to use the WGS84 ellipsoid instead of the GRS80 ellipsoid. How much difference does this make?

A first guess would be not much as the difference in the semi-minor axis (polar radius) is only 0.105 mm (0.000105 m).

For the example at hand:

	a	1/f	ESF
GRS80	6378137.000000000	298.257222100880000	0.999,870,539,895,356
WGS84	6378137.000000000	298.257223563000000	0.999,870,539,895,357

Thus, there is no difference in scale factor based on using WGS84 instead of GRS80.

Ellipsoid vs. Orthometric Height

A more common error is for the Elevation Scale Factor to be computed at the orthometric height, instead of the ellipsoid height. Several years ago, I found that a highly popular field software was using this method.

For the example at hand, the GEOID difference is 24.360 m. Computing the ESF using the ellipsoid and orthometric heights, then scaling and computing the difference over 1 mile (5280 feet) we find a 0.02' difference in computed ground distances:

	Height (M)	ESF	Difference per Mile (ft)
Ellipsoid	825.063	0.999870539895	0.6836
Orthometric	849.423	0.999866718093	0.7038

Conclusion

While the differences between all the presented ESF computation methods are very small, I am replacing my traditional method with the NGS method and will use it in future projects.